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Anomalous Hall Resistance in Bilayer Quantum Hall Systems

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We present a microscopic theory of the Hall current in the bilayer quantum Hall system on the basis of noncommutative geometry. By analyzing the Heisenberg equation of motion and the continuity equation of charge, we demonstrate the emergence of the phase current in a system where the interlayer phase coherence develops spontaneously. The phase current arranges itself to minimize the total energy of the system, as induces certain anomalous behaviors in the Hall current in the counterflow geometry and also in the drag experiment. They explain the recent experimental data for anomalous Hall resistances due to Kellogg et al. [M. Kellogg, I.B. Spielman, J.P. Eisenstein, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. **88** (2002) 126804; M. Kellogg, J.P. Eisenstein, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. **93** (2004) 036801] and Tutuc et al. [E. Tutuc, M. Shayegan and D.A. Huse, Phys. Rev. Lett. **93** (2004) 036802] at $\nu = 1$.

I. INTRODUCTION

The emergence of the Hall plateau together with the vanishing longitudinal resistance has been considered to be the unique signal of the quantum Hall (QH) effect[1,2]. This is certainly the case in the monolayer QH system. However, recent experiments[3,4] have revealed an anomalous behavior of the Hall resistance in a counterflow geometry in the bilayer QH system that both the longitudinal and Hall resistances vanish at the total bilayer filling factor $\nu = 1$. Another anomalous Hall resistance has been reported in a drag experiment[5]. Though a suggestion[6] has been made that the anomalous phenomenon would occur owing to excitonic excitations (electron-hole pairs in opposite layers) in the counterflow transport, there exists no theory demonstrating these phenomena explicitly in a unified way.

The aim of this paper is to present a microscopic theory of Hall currents to understand the mechanism of the anomalous Hall resistance[3,4,5] discovered experimentally. In the ordinary theory the current is defined as the Nöther current, which arises from the kinetic Hamiltonian. However, this is quite a nontrivial problem in the QH system[7] since the kinetic Hamiltonian is quenched within each Landau level[8,9]. There is a Lagrangian approach[10] to this problem, but it seems quite difficult to go beyond the one-body formalism within this approach. We propose a new formalism to elucidate the current in the QH system.

We start with a microscopic Hamiltonian[11] describing electrons in the lowest Landau level. The intriguing feature is that the dynamics is determined not by the kinetic Hamiltonian but by noncommutative geometry. The noncommutative geometry[12] means that the guiding center $\boldsymbol{X} = (X, Y)$ is subject to the noncommutative relation, $[X, Y] = -i\ell_B^2$, with ℓ_B the magnetic length. We derive the formula for the electric current from the Heisenberg equation of motion and the continuity equation of charge based on the noncommutative relation. It agrees with the standard formula for the Hall current in the monolayer system.

There arises new phenomena associated with the interlayer phase coherence in the bilayer QH system[1,2]. The bilayer system has the pseudospin degree of freedom, where the electron in the front (back) layer is assigned to carry the up (down) pseudospin. Provided the layer separation d is reasonably small, the interlayer phase coherence[13,14] emerges due to the Coulomb exchange interaction. The system is called the pseudospin QH ferromagnet. This is clearly seen by examining the coherence length ξ_ϑ of the interlayer phase field $\vartheta(\boldsymbol{x})$, which is calculated as

$$\xi_\vartheta = 2\ell_B \sqrt{\frac{\pi J_s^d}{\Delta_{\text{SAS}}}}, \tag{1.1}$$

where J_s^d is the pseudospin stiffness and Δ_{SAS} is the tunneling gap. It is observed that the interlayer phase coherence develops well for $J_s^d \gg \Delta_{\text{SAS}}$.

It has been argued in an effective theory[1] that the phase current, $\propto \partial_i\vartheta(\boldsymbol{x})$, flows in the pseudospin QH ferromagnet. In this paper, we present a microscopic formulation of the phase current, and show that the phase current arranges itself to minimize the total energy of the system and makes the Hall resistance vanish in a counterflow geometry[3,4]. Furthermore, it explains also the anomalous Hall resistance in the drag experiment[5].

This paper is composed as follows. Section II is devoted to a concise review of the microscopic formalism of the QH system based on the noncommutative geometry. In Section III we analyze the Heisenberg equation of motion in the QH system. In Section IV the formula is derived for the current in the spin ferromagnet from the continuity equation. In Section V we study the current in the pseudospin ferromagnet. In Section VI we determine the phase current by minimizing the total energy of the system. In Section VII we investigate how the anomalous Hall resistance occurs in the pseudospin ferromagnet.

II. NONCOMMUTATIVE GEOMETRY

A planar electron performs cyclotron motion in magnetic field, $\boldsymbol{B} = (0, 0, -B_\perp)$. The electron coordinate $\boldsymbol{x} = (x, y)$ is decomposed into the guiding center $\boldsymbol{X} = (X, Y)$ and the relative coordinate $\boldsymbol{R} = (R_x, R_y)$, $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{R}$, where $R_x = -P_y/eB_\perp$ and $R_y = P_x/eB_\perp$ with $\boldsymbol{P} = (P_x, P_y)$ the covariant momentum. The commutation relations are $[X, Y] = -i\ell_B^2$, $[P_x, P_y] = i\hbar^2/\ell_B^2$ and $[X, P_x] = [X, P_y] = [Y, P_x] = [Y, P_y] = 0$, with $\ell_B^2 = \hbar/eB_\perp$. They imply that the guiding center and the relative coordinate are independent variables.

The kinetic Hamiltonian,

$$H_{\mathrm{K}} = \frac{\boldsymbol{P}^2}{2M} = \frac{1}{2M}(P_x - iP_y)(P_x + iP_y) + \frac{1}{2}\hbar\omega_{\mathrm{c}}, \quad (2.1)$$

creates Landau levels with gap energy $\hbar\omega_{\mathrm{c}} = \hbar eB_\perp/M$. When it is large enough, excitations across Landau levels are suppressed at sufficiently low temperature. It is a good approximation to prohibit all such excitations by requiring electron confinement to the Lowest Landau level.

We explore the physics of electrons confined to the lowest Landau level, where the electron position is specified solely by the guiding center $\boldsymbol{X} = (X, Y)$, whose X and Y components are noncommutative,

$$[X, Y] = -i\ell_B^2. \quad (2.2)$$

We introduce the operators

$$b = \frac{1}{\sqrt{2}\ell_B}(X - iY), \quad b^\dagger = \frac{1}{\sqrt{2}\ell_B}(X + iY), \quad (2.3)$$

obeying $[b, b^\dagger] = 1$, and define the Fock states

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle, \qquad b|0\rangle = 0, \quad (2.4)$$

where $n = 0, 1, 2, \cdots$. The QH system provides us with an ideal 2-dimensional world with the built-in noncommutative geometry.

We assume that the electron carries the SU(N) index. For instance, it carries the spin SU(2) index in the monolayer system, and the spin-pseudospin SU(4) index in the bilayer system. The SU(N) electron field $\Psi(\boldsymbol{x})$ has N components. It is given by

$$\psi_\mu(\boldsymbol{x}) = \sum_n \langle \boldsymbol{x}|n\rangle c_\mu(n) \quad (2.5)$$

for electrons in the lowest Landau level, where $c_\mu(n)$ is the annihilation operator acting on the Fock state $|n\rangle$,

$$\{c_\mu(n), c_\nu^\dagger(m)\} = \delta_{mn}\delta_{\mu\nu}. \quad (2.6)$$

The physical variables are the electron density $\rho(\boldsymbol{x})$ and the isospin field $I_A(\boldsymbol{x})$,

$$\rho(\boldsymbol{x}) = \Psi^\dagger(\boldsymbol{x})\Psi(\boldsymbol{x}), \quad I_A(\boldsymbol{x}) = \frac{1}{2}\Psi^\dagger(\boldsymbol{x})\lambda_A\Psi(\boldsymbol{x}), \quad (2.7)$$

where λ_A are the generating matrices of SU(N). We summarize the electron density and the isospin density into the density matrix as

$$D_{\mu\nu} = \frac{1}{N}\delta_{\mu\nu}\rho + (\lambda_A)_{\mu\nu}I_A. \quad (2.8)$$

It is given by

$$D_{\mu\nu}(\boldsymbol{p}) = e^{-\ell_B^2\boldsymbol{p}^2/4}\hat{D}_{\mu\nu}(\boldsymbol{p}) \quad (2.9)$$

in the momentum space, together with

$$\hat{D}_{\mu\nu}(\boldsymbol{p}) = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{p}\boldsymbol{X}}|n\rangle c_\nu^\dagger(m)c_\mu(n). \quad (2.10)$$

We call $\hat{D}_{\mu\nu}(\boldsymbol{p})$ the bare density. The difference between $D_{\mu\nu}$ and $\hat{D}_{\mu\nu}$ is negligible for sufficiently smooth field configurations.

In the succeeding analysis of the dynamics of the SU(N) QH system, the $\mathrm{W}_\infty(\mathrm{N})$ algebra satisfied by the bare density,

$$\begin{aligned}2\pi[\hat{D}_{\mu\nu}(\boldsymbol{p}), \hat{D}_{\sigma\tau}(\boldsymbol{q})] =& \delta_{\mu\tau}e^{+\frac{i}{2}\ell_B^2\boldsymbol{p}\wedge\boldsymbol{q}}\hat{D}_{\sigma\nu}(\boldsymbol{p}+\boldsymbol{q}) \\ &- \delta_{\sigma\nu}e^{-\frac{i}{2}\ell_B^2\boldsymbol{p}\wedge\boldsymbol{q}}\hat{D}_{\mu\tau}(\boldsymbol{p}+\boldsymbol{q}),\end{aligned} \quad (2.11)$$

plays the basic role. We have already derived this relation based on the noncommutative relation (2.2) and the anticommutation relation (2.6) in our previous works[11,15]: See (3.19) in the first reference of Ref.[11].

III. EQUATIONS OF MOTION

The Heisenberg equation of motion determines the quantum mechanical system. It is given by

$$i\hbar\frac{d}{dt}\hat{D}_{\mu\nu}(\boldsymbol{p}) = [\hat{D}_{\mu\nu}(\boldsymbol{p}), H] \quad (3.1)$$

for electrons in the lowest Landau level. In ordinary physics the dynamics arises from the kinetic Hamiltonian. However, this is not the case here, since the kinetic Hamiltonian (2.1) commutes with $\hat{D}_{\mu\nu}(\boldsymbol{p})$. Indeed, H_{K} contains only the relative coordinate $\boldsymbol{R} = (-P_y/eB_\perp, P_x/eB_\perp)$, while $\hat{D}_{\mu\nu}(\boldsymbol{p})$ contains only the guiding center $\boldsymbol{X} = (X, Y)$. Nontrivial dynamics can arise because the guiding center has the noncommutative coordinates obeying (2.2). It is remarkable that the dynamics arises from the very nature of noncommutative geometry in the QH system.

The total Hamiltonian H consists of the Coulomb term H_{C} and the rest term H_{rest},

$$H = H_{\mathrm{C}} + H_{\mathrm{rest}}, \quad (3.2)$$

where H_{rest} stands for the Zeeman term in the monolayer system and additionally the tunneling and bias terms in the bilayer system. All of them are represented in terms

of the bare densities. Hence, we are able to calculate the Heisenberg equation of motion (3.1) based on the $W_\infty(N)$ algebra (2.11).

What is observed experimentally is the classical field $\mathcal{D}_{\mu\nu}(\boldsymbol{x})$, which is the expectation value of $\hat{D}_{\mu\nu}(\boldsymbol{x})$ by a Fock state,

$$\mathcal{D}_{\mu\nu}(\boldsymbol{x}) = \langle \hat{D}_{\mu\nu}(\boldsymbol{x})\rangle. \tag{3.3}$$

We consider the class of Fock states which can be written as

$$|\mathfrak{S}\rangle = e^{iW}|\mathfrak{S}_0\rangle, \tag{3.4}$$

where W is an arbitrary element of the $W_\infty(N)$ algebra which represents a general linear combination of the operators $c_\nu^\dagger(m)c_\mu(n)$. The state $|\mathfrak{S}_0\rangle$ is assumed to be of the form[11]

$$|\mathfrak{S}_0\rangle = \prod_{\mu,n}\left[c_\mu^\dagger(n)\right]^{\nu_\mu(n)}|0\rangle, \tag{3.5}$$

where $\nu_\mu(n)$ takes the value either 0 or 1 specifying whether the isospin state μ at the state $|n\rangle$ is occupied or not, respectively. Though it may not include all states, it certainly contains all integer QH states, by which we mean the ground state as well as all quasiparticle excited states at ν =integer.

The classical field satisfied the classical equation of motion. It is constructed by taking the expectation value of the Heisenberg equation of motion (3.1),

$$i\hbar\frac{d}{dt}\mathcal{D}_{\mu\nu}(\boldsymbol{x}) = \langle \hat{D}_{\mu\nu}(\boldsymbol{p})H\rangle - \langle H\hat{D}_{\mu\nu}(\boldsymbol{p})\rangle. \tag{3.6}$$

We can verify[11] for the class of states (3.4) that

$$\frac{d}{dt}\mathcal{D}_{\mu\nu}(\boldsymbol{x}) = [\mathcal{D}_{\mu\nu}(\boldsymbol{x}), \mathcal{H}]_{\mathrm{PB}}, \tag{3.7}$$

where $\mathcal{H} = \langle H\rangle$ is the classical Hamiltonian, provided the classical density is endowed with the Poisson structure

$$\begin{aligned}2\pi i\hbar[\mathcal{D}_{\mu\nu}(\boldsymbol{p}), \mathcal{D}_{\sigma\tau}(\boldsymbol{q})]_{\mathrm{PB}} =&\delta_{\mu\tau}e^{+\frac{i}{2}\ell_B^2\boldsymbol{p}\wedge\boldsymbol{q}}\mathcal{D}_{\sigma\nu}(\boldsymbol{p}+\boldsymbol{q})\\ &- \delta_{\sigma\nu}e^{-\frac{i}{2}\ell_B^2\boldsymbol{p}\wedge\boldsymbol{q}}\mathcal{D}_{\mu\tau}(\boldsymbol{p}+\boldsymbol{q}).\end{aligned} \tag{3.8}$$

The classical Coulomb energy consists of the direct and exchange energies[11], $\mathcal{H}_{\mathrm{C}} = \mathcal{H}_{\mathrm{D}} + \mathcal{H}_{\mathrm{X}}$, and we obtain

$$\mathcal{H} = \mathcal{H}_{\mathrm{D}} + \mathcal{H}_{\mathrm{X}} + \mathcal{H}_{\mathrm{rest}}, \tag{3.9}$$

from the total Hamiltonian (3.2). Here, $\mathcal{H}_{\mathrm{D}}$ and $\mathcal{H}_{\mathrm{rest}}$ have the same expressions as H_{C} and H_{rest}, respectively, with the replacement of $\hat{D}_{\mu\nu}$ by $\mathcal{D}_{\mu\nu}$: The exchange term $\mathcal{H}_{\mathrm{X}}$ is a new term. We give explicit expressions of these terms in the following sections.

IV. CURRENTS IN SPIN QH FERROMAGNET

In the monolayer QH system the physical variables are the electron density $\rho(\boldsymbol{x})$ and the spin field $S_a(\boldsymbol{x})$,

$$\rho(\boldsymbol{x}) = \Psi^\dagger(\boldsymbol{x})\Psi(\boldsymbol{x}), \quad S_a(\boldsymbol{x}) = \frac{1}{2}\Psi^\dagger(\boldsymbol{x})\tau_a\Psi(\boldsymbol{x}) \tag{4.1}$$

with (2.5) for $\Psi(\boldsymbol{x})$. We denote the corresponding classical field as

$$\varrho(\boldsymbol{x}) = \langle\hat{\rho}(\boldsymbol{x})\rangle, \quad \mathcal{S}(\boldsymbol{x}) = \langle\hat{S}(\boldsymbol{x})\rangle. \tag{4.2}$$

Here τ_a are the Pauli matrices for the spin space.

We study the electric current in the QH state. The current is introduced originally to guarantee the charge conservation. This is the case also in the noncommutative plane,

$$-e\frac{d}{dt}\hat{\rho}(\boldsymbol{x}) = \partial_i\hat{J}_i(\boldsymbol{x}), \tag{4.3}$$

where $-e\hat{\rho}(\boldsymbol{x})$ is the charge density and $\hat{J}_i(\boldsymbol{x})$ is the current in the lowest Landau level. The physically observed current is the classical current given by

$$\mathcal{J}_i(\boldsymbol{x}) = \langle\hat{J}_i(\boldsymbol{x})\rangle. \tag{4.4}$$

Taking the expectation value of (4.3) and using the classical equation of motion (3.7), we have

$$\partial_i\mathcal{J}_i(\boldsymbol{x}) = -e\frac{d}{dt}\varrho(\boldsymbol{x}) = -e[\varrho(\boldsymbol{x}), \mathcal{H}]_{\mathrm{PB}}. \tag{4.5}$$

The formula for the current $\mathcal{J}_i(\boldsymbol{x})$ is obtained from this continuity equation by integrating it.

In the monolayer QH system the Hamiltonian consists of the Coulomb term, the Zeeman term and the electric-field term,

$$H = H_{\mathrm{C}} + H_{\mathrm{Z}} + H_{\mathrm{E}}. \tag{4.6}$$

We introduce the scalar potential $\varphi(\boldsymbol{x})$ to produce an electric field,

$$E_i(\boldsymbol{x}) = -\partial_i\varphi(\boldsymbol{x}), \tag{4.7}$$

in the Hamiltonian H_{E}. The classical Hamiltonian is given by $\mathcal{H} = \langle H\rangle$, which reads[11]

$$\mathcal{H} = \mathcal{H}_{\mathrm{D}} + \mathcal{H}_{\mathrm{X}} + \mathcal{H}_{\mathrm{Z}} + \mathcal{H}_{\mathrm{E}}, \tag{4.8}$$

where

$$\mathcal{H}_{\mathrm{D}} =\pi\int d^2q\, V_{\mathrm{D}}(\boldsymbol{q})\varrho(-\boldsymbol{q})\varrho(\boldsymbol{q}), \tag{4.9a}$$

$$\begin{aligned}\mathcal{H}_{\mathrm{X}} =& -\pi\int d^2p\, V_{\mathrm{X}}(\boldsymbol{p})\Big[\sum_{a=xyz}\mathcal{S}_a(-\boldsymbol{p})\mathcal{S}_a(\boldsymbol{p})\\ &+\frac{1}{4}\varrho(-\boldsymbol{p})\varrho(\boldsymbol{p})\Big],\end{aligned} \tag{4.9b}$$

$$\mathcal{H}_{\mathrm{Z}} = -2\pi\Delta_{\mathrm{Z}}\mathcal{S}_z(0). \tag{4.9c}$$

$$\mathcal{H}_{\mathrm{E}} = -e\int d^2q\, e^{-\boldsymbol{q}^2\ell_B^2/4}\varphi(-\boldsymbol{q})\varrho(\boldsymbol{q}), \tag{4.9d}$$

with

$$V_{\rm D}(\boldsymbol{q}) =\frac{e^2}{4\pi\varepsilon|\boldsymbol{q}|}e^{-\ell_B^2\boldsymbol{q}^2/2}, \tag{4.10}$$

$$V_{\rm X}(\boldsymbol{p}) =\frac{\sqrt{2\pi}e^2\ell_B}{4\pi\varepsilon}I_0(\ell_B^2\boldsymbol{p}^2/4)e^{-\ell_B^2\boldsymbol{p}^2/4}. \tag{4.11}$$

Here, $I_0(x)$ is the modified Bessel function.

It is straightforward to calculate the Poisson bracket (4.5) with (4.8),

$$[\varrho(\boldsymbol{k}),\mathcal{H}_{\rm D}]_{\rm PB} =\frac{2}{\hbar}\int d^2q\, V_{\rm D}(\boldsymbol{q})\{\varrho(\boldsymbol{q}),\varrho(\boldsymbol{k}-\boldsymbol{q})\}\sin\left(\ell_B^2\frac{\boldsymbol{k}\wedge\boldsymbol{q}}{2}\right), \tag{4.12a}$$

$$[\varrho(\boldsymbol{k}),\mathcal{H}_{\rm X}]_{\rm PB} =-\frac{1}{2\hbar}\int d^2q\, V_{\rm X}(\boldsymbol{q})\varrho(-\boldsymbol{q})\varrho(\boldsymbol{k}+\boldsymbol{q})\sin\left(\ell_B^2\frac{\boldsymbol{k}\wedge\boldsymbol{q}}{2}\right) -\frac{2}{\hbar}\int d^2q\, V_{\rm X}(\boldsymbol{q})S_a(-\boldsymbol{q})S_a(\boldsymbol{k}+\boldsymbol{q})\sin\left(\ell_B^2\frac{\boldsymbol{k}\wedge\boldsymbol{q}}{2}\right) \tag{4.12b}$$

$$[\varrho(\boldsymbol{k}),\mathcal{H}_{\rm Z}]_{\rm PB}=0, \tag{4.12c}$$

$$[\varrho(\boldsymbol{k}),\mathcal{H}_{\rm E}]_{\rm PB} =-\frac{e}{2\pi\hbar}\int d^2q\, e^{-\boldsymbol{q}^2\ell_B^2/4}\varphi(\boldsymbol{q})\varrho(\boldsymbol{k}-\boldsymbol{q})\sin\left(\ell_B^2\frac{\boldsymbol{k}\wedge\boldsymbol{q}}{2}\right). \tag{4.12d}$$

We evaluate them on the ground state in the QH system, where the classical density is given by

$$\varrho(\boldsymbol{k})=2\pi\rho_0\delta^2(\boldsymbol{k}) \tag{4.13}$$

with ρ_0 the total electron density. This is known as the incompressibility condition[1], implying that the QH system is an incompressible liquid.

We first examine the Poisson bracket (4.12b) for $\mathcal{H}_{\rm D}$. Substituting the incompressibility condition into (4.12a) it is trivial to see that $[\varrho(\boldsymbol{k}),\mathcal{H}_{\rm D}]_{\rm PB}=0$. Hence, there is no contribution to the Hall current from the direct Coulomb term $\mathcal{H}_{\rm D}$.

We next examine the Poisson bracket (4.12b) for $\mathcal{H}_{\rm X}$. The term involving $\varrho(-\boldsymbol{k}')\varrho(\boldsymbol{k}+\boldsymbol{k}')\sin[\ell_B^2(\boldsymbol{k}\wedge\boldsymbol{q})/2]$ vanishes because of the incompressibility condition. The remaining term involves $S_z(-\boldsymbol{k}')S_z(\boldsymbol{k}+\boldsymbol{k}')$. Since we are concerned about a homogeneous flow of electrons, taking the nontrivial lowest order term in the derivative expansion of potential $V_{\rm X}(\boldsymbol{k})$, we find

$$\begin{aligned}&\int d^2q\, V_{\rm X}(\boldsymbol{k}')S_z(-\boldsymbol{q})S_z(\boldsymbol{k}+\boldsymbol{q})\sin\left(\frac{1}{2}\ell_B^2\boldsymbol{k}\wedge\boldsymbol{q}\right)\\ \simeq&V_{\rm X}(0)\int d^2q\, S_z(-\boldsymbol{q})S_z(\boldsymbol{k}+\boldsymbol{q})\sin\left(\frac{1}{2}\ell_B^2\boldsymbol{k}\wedge\boldsymbol{q}\right)\\ =&0.\end{aligned} \tag{4.14}$$

This is zero because the relation

$$\int d^2q\, f(-\boldsymbol{q})g(\boldsymbol{k}+\boldsymbol{q})\sin(\boldsymbol{k}\wedge\boldsymbol{q}) =-\int d^2q\, f(\boldsymbol{k}+\boldsymbol{q})g(-\boldsymbol{q})\sin(\boldsymbol{k}\wedge\boldsymbol{q}) \tag{4.15}$$

holds for any two functions f and g. Hence, there is no contribution from $\mathcal{H}_{\rm X}$.

We finally examine the contribution from $\mathcal{H}_{\rm E}$ by evaluating (4.12d). Expanding $\sin[\ell_B^2(\boldsymbol{k}\wedge\boldsymbol{q})/2]$, we obtain

$$\mathcal{J}_i(\boldsymbol{k}) =-i\frac{e^2\ell_B^2}{2\pi\hbar}\varepsilon_{ij}\int d^2q\, e^{-\boldsymbol{k}^2\ell_B^2/4}q_j\varphi(\boldsymbol{q})\varrho(\boldsymbol{k}-\boldsymbol{q}) \times\left[1-\frac{1}{3!}\left(\ell_B^2\frac{\boldsymbol{k}\wedge\boldsymbol{q}}{2}\right)^2+\cdots\right]. \tag{4.16}$$

In a constant electric field E_j such that

$$q_i\varphi(\boldsymbol{q})=2\pi iE_j\delta(\boldsymbol{q}), \tag{4.17}$$

we find

$$\mathcal{J}_i(\boldsymbol{k})=\frac{e^2}{\hbar}\ell_B^2\varepsilon_{ij}E_je^{-\boldsymbol{k}^2\ell_B^2/4}\varrho(\boldsymbol{k}-\boldsymbol{q}). \tag{4.18}$$

On the incompressible state (4.13) it yields

$$\mathcal{J}_i(\boldsymbol{x})=\frac{e^2\ell_B^2}{\hbar}\varepsilon_{ij}E_j\rho_0. \tag{4.19}$$

This is the standard formula for the Hall current.

We have demonstrated that the Coulomb and Zeeman interactions do not affect the Hall current, as expected. However, this is not a trivial result since the exchange Coulomb interaction yields rather complicated formulas in the midstream of calculations. As we have verified, they vanish in the spin QH ferromagnet. On the other hand, as we shall see in the succeeding sections, the exchange Coulomb interaction produces the phase current in the pseudospin QH ferromagnet.

V. CURRENTS IN PSEUDOSPIN QH FERROMAGNET

We proceed to study the electric currents in the bilayer system. Though the actual system has the spin-pseudospin SU(4) structure, we consider the spin-frozen system since the spin does not affect the current: See Appendix B. The electron field (2.5) has two components $\psi^\alpha(\boldsymbol{x})$ corresponding to the front (α=f) and back (α=b) layers. The physical variables are the electron densities $\rho^\alpha(\boldsymbol{x})$ in the two layers, and the pseudospin field $P_a(\boldsymbol{x})$,

$$\rho^\alpha(\boldsymbol{x})=\psi^{\alpha\dagger}(\boldsymbol{x})\psi^\alpha(\boldsymbol{x}),\quad P_a(\boldsymbol{x})=\frac{1}{2}\Psi^\dagger(\boldsymbol{x})\pi_a\Psi(\boldsymbol{x}) \tag{5.1}$$

with (2.5), where π_a are the Pauli matrices for the pseudospin space. We use notations

$$\varrho^\alpha(\boldsymbol{x})=\langle\hat\rho^\alpha(\boldsymbol{x})\rangle,\quad \mathcal{P}_a(\boldsymbol{x})=\langle\hat P_a(\boldsymbol{x})\rangle \tag{5.2}$$

for the classical variables.

The Hamiltonian H consists of the Coulomb term, the tunneling term, the gate term and the electric-field term,

$$H = H_{\rm C} + H_{\rm T} + H_{\rm gate} + H_{\rm E}, \tag{5.3}$$

which are explicitly given by (A1) in Appendix A. The gate term has been introduced to make a density imbalance between the two layers. The average density reads

$$\rho_0^{\rm f} = \frac{1+\sigma_0}{2}\rho_0, \quad \rho_0^{\rm b} = \frac{1-\sigma_0}{2}\rho_0 \tag{5.4}$$

in each layer. We call σ_0 the imbalance parameter.

As we show in Appendix A, the classical Hamiltonian $\mathcal{H}$ is rearranged into

$$\mathcal{H} \equiv \langle H\rangle = \mathcal{H}_{\rm D} + \mathcal{H}_{\rm X} + \mathcal{H}_{\rm T} + \mathcal{H}_{\rm bias} + \mathcal{H}_{\rm E}, \tag{5.5}$$

where $\mathcal{H}_{\rm D}$ and $\mathcal{H}_{\rm X}$ are the direct and exchange Coulomb energy terms,

$$\begin{aligned}\mathcal{H}_{\rm D} =& \pi\int d^2p V_{\rm D}^{+}(\boldsymbol{p})\varrho(-\boldsymbol{p})\varrho(\boldsymbol{p}) \\ &+ 4\pi\int d^2p V_{\rm D}^{-}(\boldsymbol{p})\mathcal{P}_z(-\boldsymbol{p})\mathcal{P}_z(\boldsymbol{p}) - 8\pi\epsilon_{\rm D}^{-}\sigma_0\mathcal{P}_z(0)\end{aligned} \tag{5.6a}$$

$$\begin{aligned}\mathcal{H}_{\rm X} =& -\pi\sum_{a=x,y}\int d^2p V_{\rm X}^{d}(\boldsymbol{p})\mathcal{P}_a(-\boldsymbol{p})\mathcal{P}_a(\boldsymbol{p}) \\ &- \pi\int d^2p V_{\rm X}(\boldsymbol{p})\left[\mathcal{P}_z(-\boldsymbol{p})\mathcal{P}_z(\boldsymbol{p}) + \frac{1}{4}\varrho(-\boldsymbol{p})\varrho(\boldsymbol{p})\right] \\ &+ 8\pi\epsilon_{\rm X}^{-}\sigma_0\mathcal{P}_z(0),\end{aligned} \tag{5.6b}$$

and $\mathcal{H}_{\rm T}$ and $\mathcal{H}_{\rm bias}$ are the tunneling and bias terms,

$$\mathcal{H}_{\rm T} = -2\pi\Delta_{\rm SAS}P_x(0), \tag{5.6c}$$

$$\mathcal{H}_{\rm bias} = -2\pi\frac{\sigma_0}{\sqrt{1-\sigma_0^2}}\Delta_{\rm SAS}P_z(0), \tag{5.6d}$$

while the electric-field term reads

$$\mathcal{H}_{\rm E} = -e\int d^2q\, e^{-\boldsymbol{q}^2\ell_B^2/4}\left[\varphi^{\rm f}(-\boldsymbol{q})\varrho^{\rm f}(\boldsymbol{q}) + \varphi^{\rm b}(-\boldsymbol{q})\varrho^{\rm b}(\boldsymbol{q})\right]. \tag{5.6e}$$

Various Coulomb potentials are defined by

$$V_{\rm X} = V_{\rm X}^{+} + V_{\rm X}^{-}, \qquad V_{\rm X}^{d} = V_{\rm X}^{+} - V_{\rm X}^{-}, \tag{5.7}$$

and

$$V_{\rm D}^{\pm}(\boldsymbol{p}) = \frac{e^2}{8\pi\varepsilon|\boldsymbol{p}|}\left(1\pm e^{-|\boldsymbol{p}|d}\right)e^{-\frac{1}{2}\ell_B^2\boldsymbol{p}^2}, \tag{5.8a}$$

$$V_{\rm X}^{\pm}(\boldsymbol{p}) = \frac{\ell_B^2}{\pi}\int d^2k\, e^{-i\ell_B^2\boldsymbol{p}\wedge\boldsymbol{k}}V_{\rm D}^{\pm}(\boldsymbol{k}), \tag{5.8b}$$

with the interlayer separation d. We have also defined

$$\epsilon_{\rm D}^{-} = \frac{1}{2}\rho_0\int d^2x\, V_{\rm D}^{-}(\boldsymbol{x}), \quad \epsilon_{\rm X}^{-} = \frac{1}{4}\rho_0\int d^2x\, V_{\rm X}^{-}(\boldsymbol{x}). \tag{5.9}$$

We note that the capacitance energy is given by

$$\epsilon_{\rm cap} = 4(\epsilon_{\rm D}^{-} - \epsilon_{\rm X}^{-}). \tag{5.10}$$

The scalar potentials $\varphi^{\rm f}(\boldsymbol{x})$ and $\varphi^{\rm b}(\boldsymbol{x})$ are introduced to produce the electric fields

$$E_i^{\rm f} = -\partial_i\varphi^{\rm f}, \qquad E_i^{\rm b} = -\partial_i\varphi^{\rm b} \tag{5.11}$$

in the front and back layers within the Hamiltonian (5.6e).

We investigate the classical equation of motion (3.7) to derive the classical current $\mathcal{J}_i^{\alpha}(\boldsymbol{x})$ in each layer. It is defined so that the charge conserves locally,

$$-e\frac{d\varrho^{\rm f}}{dt} = \partial_i\mathcal{J}_i^{\rm f}(\boldsymbol{x}) - \frac{1}{d}\mathcal{J}_z(\boldsymbol{x}), \tag{5.12a}$$

$$-e\frac{d\varrho^{\rm b}}{dt} = \partial_i\mathcal{J}_i^{\rm b}(\boldsymbol{x}) + \frac{1}{d}\mathcal{J}_z(\boldsymbol{x}), \tag{5.12b}$$

where $\mathcal{J}_z(\boldsymbol{x})$ is the tunneling current between the two layers.

The tunneling term $\mathcal{H}_{\rm T}$ contributes only to the tunneling current $\mathcal{J}_z(\boldsymbol{x})$, and it is given by

$$\frac{1}{d}\mathcal{J}_z(\boldsymbol{x}) = \frac{e}{2}\frac{d}{dt}[\varrho^{\rm f} - \varrho^{\rm b}]\Big|_{\mathcal{H}_{\rm T}} = e[\mathcal{P}_z, \mathcal{H}_{\rm T}]_{\rm PB}. \tag{5.13}$$

The current in the layer $\alpha =$f, b is given by

$$\partial_i\mathcal{J}_i^{\alpha}(\boldsymbol{x}) = -e\left.\frac{d\varrho^{\alpha}}{dt}\right|_{\mathcal{H}_{\rm D}+\mathcal{H}_{\rm X}+\mathcal{H}_{\rm bias}+\mathcal{H}_{\rm E}}, \tag{5.14}$$

which consists of $[\varrho^{\alpha}, \mathcal{H}_{\rm D}]_{\rm PB}$, $[\varrho^{\alpha}, \mathcal{H}_{\rm X}]_{\rm PB}$, $[\varrho^{\alpha}, \mathcal{H}_{\rm bias}]_{\rm PB}$ and $[\varrho^{\alpha}, \mathcal{H}_{\rm E}]_{\rm PB}$. We express ϱ^{α} in terms of ϱ and $\mathcal{P}_z$ as

$$\varrho^{\rm f} = \frac{1}{2}\varrho + \mathcal{P}_z, \qquad \varrho^{\rm b} = \frac{1}{2}\varrho - \mathcal{P}_z. \tag{5.15}$$

Note that $\varrho^{\alpha} = \rho_0^{\alpha}$ with (5.4) in the ground state.

We need to calculate the Poisson brackets for ϱ and $\mathcal{P}_z$ with various Hamiltonians. When we estimate them on the state satisfying the incompressibility condition (4.13), many terms vanish precisely by the same reasons as in the monolayer system. We now demonstrate that there exists a new contribution from the exchange interaction $\mathcal{H}_{\rm X}$ to the current, as is the novel feature in the pseudospin QH ferromagnet. Let us explicitly write down only those parts in the Poisson brackets that yield nonvanishing contributions to the current.

There is a new term involving $\cos(\frac{1}{2}\ell_B^2\boldsymbol{k}\wedge\boldsymbol{q})$ in the Poisson bracket $[\mathcal{P}_z(\boldsymbol{k}), \mathcal{H}_{\rm X}]_{\rm PB}$,

$$\begin{aligned}&[\mathcal{P}_z(\boldsymbol{k}), \mathcal{H}_{\rm X}]_{\rm PB} \\ =& -\frac{\epsilon_{ab}}{\hbar}\int d^2q\, V_{\rm X}^{d}(\boldsymbol{q})\mathcal{P}_a(-\boldsymbol{q})\mathcal{P}_b(\boldsymbol{k}+\boldsymbol{q})\cos\left(\frac{\ell_B^2}{2}\boldsymbol{k}\wedge\boldsymbol{q}\right) \\ &+\cdots,\end{aligned} \tag{5.16}$$

where the index a runs over x and y. Making the derivative expansion of $V_{\rm X}^d(\boldsymbol{q})$ and $\cos(\frac{1}{2}\ell_B^2\boldsymbol{k}\wedge\boldsymbol{q})$, we find

$$\mathcal{J}_i^{\rm f(X)}(\boldsymbol{x})=-\mathcal{J}_i^{\rm b(X)}(\boldsymbol{x})=\frac{4eJ_s^d}{\hbar\rho_0^2}(\partial_i\mathcal{P}_x\cdot\mathcal{P}_y-\partial_i\mathcal{P}_y\cdot\mathcal{P}_x), \tag{5.17}$$

where

$$J_s^d=J_s\left[-\sqrt{\frac{2}{\pi}}\frac{d}{\ell_B}+\left(1+\frac{d^2}{\ell_B^2}\right)e^{d^2/2\ell_B^2}\mathrm{erfc}\left(d/\sqrt{2}\ell_B\right)\right] \tag{5.18}$$

together with

$$J_s=\frac{1}{16\sqrt{2\pi}}\frac{e^2}{4\pi\varepsilon\ell_B} \tag{5.19}$$

is the pseudospin stiffness.

The electric field yields a nonzero contribution as in the monolayer case,

$$\begin{aligned}&[\varrho^\alpha(\boldsymbol{k}),\mathcal{H}_{\rm E}]_{\rm PB}\\&=-\frac{e}{2\pi\hbar}\int d^2q\, e_0^{-\frac{1}{4}\ell_B^2q}\varphi^\alpha(\boldsymbol{q})\varrho^\alpha(\boldsymbol{k}-\boldsymbol{q})\sin\left(\ell_B^2\frac{\boldsymbol{k}\wedge\boldsymbol{q}}{2}\right),\end{aligned}$$

as corresponds to the monolayer formula (4.19). Hence, we obtain the standard formula for the Hall current in each layer,

$$\mathcal{J}_i^{\alpha(\rm E)}(\boldsymbol{x})=\frac{e^2\ell_B^2}{\hbar}\varepsilon_{ij}E_j\rho_0^\alpha, \tag{5.20}$$

on the incompressible ground state.

Finally, the Poisson bracket with the tunneling Hamiltonian is exactly calculable,

$$[P_y(\boldsymbol{k}),\mathcal{H}_{\rm T}]_{\rm PB}=\frac{1}{\hbar}\Delta_{\rm SAS}P_z(\boldsymbol{k}), \tag{5.21}$$

which yields

$$\mathcal{J}_z(\boldsymbol{x})=-\frac{ed}{\hbar}\Delta_{\rm SAS}P_y(\boldsymbol{x}) \tag{5.22}$$

to the tunneling current.

We parametrize the classical pseudospin field in terms of the interlayer phase field $\vartheta(\boldsymbol{x})$ and the imbalance field $\sigma(\boldsymbol{x})$,

$$\begin{aligned}P_x(\boldsymbol{x})&=\frac{1}{2}\rho_0\sqrt{1-\sigma^2(\boldsymbol{x})}\cos\vartheta(\boldsymbol{x}),\\P_y(\boldsymbol{x})&=-\frac{1}{2}\rho_0\sqrt{1-\sigma^2(\boldsymbol{x})}\sin\vartheta(\boldsymbol{x}),\\P_z(\boldsymbol{x})&=\frac{1}{2}\rho_0\sigma(\boldsymbol{x}).\end{aligned} \tag{5.23}$$

The Hall current is given by the sum of (5.17) and (5.20),

$$\mathcal{J}_i^{\rm f}(\boldsymbol{x})=\frac{eJ_s^d}{\hbar}(1-\sigma^2)\partial_i\vartheta+\frac{e^2\ell_B^2}{\hbar}\varepsilon_{ij}E_j^{\rm f}\varrho^{\rm f}, \tag{5.24a}$$

$$\mathcal{J}_i^{\rm b}(\boldsymbol{x})=-\frac{eJ_s^d}{\hbar}(1-\sigma^2)\partial_i\vartheta+\frac{e^2\ell_B^2}{\hbar}\varepsilon_{ij}E_j^{\rm b}\varrho^{\rm b}. \tag{5.24b}$$

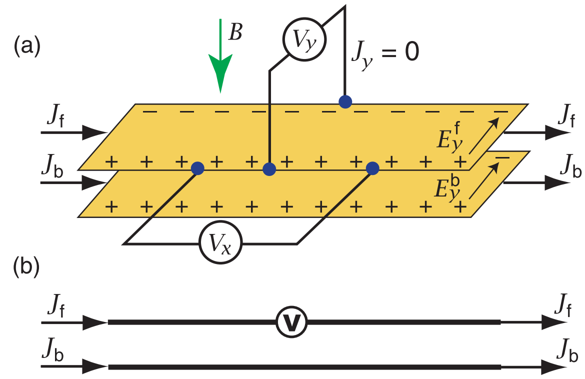

FIG. 1: (a) Hall currents are injected to the front and back layers independently. The Hall and diagonal resistances are measured only in one of the layers. (b) A simplified picture is given to represent the same measurement, where the symbol V in a circle indicates that the measurement is done on this layer.

We have shown that the phase current arises in the presence of the interlayer phase coherence.

We should mention that the emergence of the phase current, $\propto\partial_i\vartheta(\boldsymbol{x})$, in the pseudospin QH ferromagnet has already been pointed out in an effective theory[1] based on an intuitive and phenomenological reasoning. In this paper we have presented a microscopic formulation of the phase current.

VI. DIAGONAL AND HALL RESISTANCES

Let us first review the Hall current in the monolayer system with homogeneous electron density ρ_0. The electric field $\boldsymbol{E}$ drives the Hall current into the direction perpendicular to it,

$$\mathcal{J}_i=\frac{e^2\ell_B^2\rho_0}{\hbar}\varepsilon_{ij}E_j. \tag{6.1}$$

We apply the electric field so that the current flows into the x direction, as implies $E_x=0$. Hence the diagonal resistance vanishes,

$$R_{xx}\equiv\frac{E_x}{\mathcal{J}_x}=0, \tag{6.2}$$

and the Hall resistance is given by

$$R_{xy}\equiv\frac{E_y}{\mathcal{J}_x}=\frac{\hbar}{e^2\ell_B^2\rho_0}=\frac{2\pi\hbar}{\nu e^2}. \tag{6.3}$$

The signals of the QH effect consist of the dissipationless current (6.2) and the development of the Hall plateau at the magic filling factor ν.

What occurs in actual systems is as follows. We feed the current $\mathcal{J}_x$ into the x direction. Due to the Lorentz force electrons accumulate at the edge of the sample, which generates such an electric field E_y that makes

the given amount of current $\mathcal{J}_x$ flow into the x direction [Fig.1]. The relation between the current and the electric field is fixed kinematically by the formula (6.1) in the monolayer system.

We proceed to study the QH current in the imbalanced bilayer system at σ_0, where the electron densities are given by (5.4) in the ground state. We assume the sample parameter $\Delta_{\text{SAS}} = 0$ so that there is no tunneling current between the two layers. As we show in Appendix A, the interlayer phase field $\vartheta(\boldsymbol{x})$ is gapless in the limit $\Delta_{\text{SAS}} = 0$, but the imbalance field $\sigma(\boldsymbol{x})$ has the gap ϵ_{cap}. Consequently, the excitation of $\sigma(\boldsymbol{x})$ is suppressed at sufficiently low temperature. Hence we set $\sigma(\boldsymbol{x}) = \sigma_0$ in all formulas.

We imagine the electric fields E_j^{f} and E_j^{b} driving the Hall currents to flow into the x direction [Fig.1]. As we have argued in the previous section, the basic formula for the current is (5.24), or

$$\mathcal{J}_i^{\text{f}} = \frac{e}{\hbar}(1-\sigma_0^2)J_s^d \partial_i \vartheta + \frac{e^2\ell_B^2}{\hbar}\varepsilon_{ij}E_j^{\text{f}}\rho_0^{\text{f}}, \tag{6.4a}$$

$$\mathcal{J}_i^{\text{b}} = -\frac{e}{\hbar}(1-\sigma_0^2)J_s^d \partial_i \vartheta + \frac{e^2\ell_B^2}{\hbar}\varepsilon_{ij}E_j^{\text{b}}\rho_0^{\text{b}}, \tag{6.4b}$$

in the imbalance configuration at σ_0.

Since our system is assumed to be homogeneous in the y direction, the variables depend only on x. Thus,

$$E_x^{\text{f}} = E_x^{\text{b}} = 0. \qquad \partial_y \vartheta = 0, \tag{6.5}$$

and

$$R_{xx}^{\text{f}} \equiv \frac{E_x^{\text{f}}}{\mathcal{J}_x^{\text{f}}} = 0, \qquad R_{xx}^{\text{b}} \equiv \frac{E_x^{\text{b}}}{\mathcal{J}_x^{\text{b}}} = 0, \tag{6.6}$$

for $\mathcal{J}_x^{\text{f}} \neq 0$ and $\mathcal{J}_x^{\text{b}} \neq 0$. The Hall current is given by

$$\mathcal{J}_x^{\text{f}} = \frac{e}{\hbar}(1-\sigma_0^2)J_s^d \partial_x \vartheta + \frac{e^2\ell_B^2\rho_0}{2\hbar}(1+\sigma_0)E_y^{\text{f}}, \tag{6.7a}$$

$$\mathcal{J}_x^{\text{b}} = -\frac{e}{\hbar}(1-\sigma_0^2)J_s^d \partial_x \vartheta + \frac{e^2\ell_B^2\rho_0}{2\hbar}(1-\sigma_0)E_y^{\text{f}}. \tag{6.7b}$$

Consequently the relation between the current and the electric field is not fixed kinematically in the presence of the interlayer phase difference $\vartheta(\boldsymbol{x})$.

Any set of E_y^{f}, E_y^{b} and ϑ seems to yield the given amounts of currents $\mathcal{J}_x^{\text{f}}$ and $\mathcal{J}_x^{\text{b}}$ provided they satisfy (6.7). In the actual system the unique set of them is realized: It is the one that minimizes the energy of the system. It is a dynamical problem how E_y^{f} and E_y^{b} are determined in the bilayer system.

A bilayer system consists of the two layers and the volume between them. The dynamics of electrons is described by the Hamiltonian (5.5), which is defined on the two planes. The tunneling term has been introduced just to guarantee the charge conservation. We have so far neglected the electric field in the volume between the two layers, since it does not contribute to the equation of motion (3.7) for electrons. We now need to analyze the equation of motion also for the electric field. Equivalently it is necessary to minimize the Coulomb energy stored in the volume between the two layers.

The energy due to the electric field $\boldsymbol{E}(\boldsymbol{x}, z)$ between the two layers is given by the sum of the Maxwell term and the source term,

$$H_{\text{E}} = \frac{\varepsilon}{2}\int d^2x dz\, \boldsymbol{E}^2(\boldsymbol{x}, z) - e\int d^2x dz\, \varphi(\boldsymbol{x}, z)\rho(\boldsymbol{x}, z), \tag{6.8}$$

where

$$E_x = -\partial_x \varphi, \quad E_y = \partial_y \varphi, \quad E_z = -\frac{\varphi^{\text{f}} - \varphi^{\text{b}}}{d}. \tag{6.9}$$

Note that (6.8) is equivalent to the "surface term" (5.6e) in the equation of motion (3.7) since the electron density $\rho(\boldsymbol{x}, z)$ is nonzero only on the two layers.

When there are constant fields E_y^{f} and E_y^{b} into the y direction on the layers, the field E_z between the two layers is given by

$$E_z = -\frac{\varphi^{\text{f}} - \varphi^{\text{b}}}{d} = \frac{E_y^{\text{f}} - E_y^{\text{b}}}{d} y. \tag{6.10}$$

We carry out the integration over z and then over the plane in (6.8),

$$\begin{aligned} H_{\text{E}} =& \frac{\varepsilon d_w}{2}\int d^2x\, \left((E_y^{\text{f}})^2 + (E_y^{\text{b}})^2\right) + \frac{\varepsilon d}{2}\int d^2x\, E_z^2 \\ =& \frac{\varepsilon d_w L^2}{2}\left((E_y^{\text{f}})^2 + (E_y^{\text{b}})^2\right) + \frac{\varepsilon L^4}{24d}(E_y^{\text{f}} - E_y^{\text{b}})^2, \end{aligned} \tag{6.11}$$

where d_w is the thickness of the layer, and L is the size of the sample. Note that there is no contribution from the source term due to the parity,

$$\int d^2x dz\, \varphi(\boldsymbol{x}, z)\rho(\boldsymbol{x}, z) \propto \int_{-L/2}^{L/2} dy\, y = 0. \tag{6.12}$$

The energy density is given by

$$\mathcal{H}_{\text{E}} = \frac{\varepsilon d_w}{2}\left((E_y^{\text{f}})^2 + (E_y^{\text{b}})^2\right) + \frac{\varepsilon L^2}{24d}(E_y^{\text{f}} - E_y^{\text{b}})^2. \tag{6.13}$$

The important observation is that the second term diverges in the large limit of the sample size L. The order of the sample size is $L \simeq 1$ mm, while the typical size parameter is $\ell_B \simeq d \simeq d_B \simeq 10$ nm, as implies $L/\ell_B \simeq 10^6$. It is a good approximation to take the limit $L \to \infty$. We then find

$$E_y^{\text{f}} = E_y^{\text{b}} \tag{6.14}$$

to make the energy density finite.

We rewrite (6.7) as

$$E_y^{\text{f}} = \frac{2\hbar}{e^2\ell_B^2\rho_0}\left[\frac{\mathcal{J}_x^{\text{f}}}{1+\sigma_0} - \frac{e}{\hbar}(1-\sigma_0)J_s^d \partial_x \vartheta\right], \tag{6.15a}$$

$$E_y^{\text{b}} = \frac{2\hbar}{e^2\ell_B^2\rho_0}\left[\frac{\mathcal{J}_x^{\text{b}}}{1-\sigma_0} + \frac{e}{\hbar}(1+\sigma_0)J_s^d \partial_x \vartheta\right]. \tag{6.15b}$$

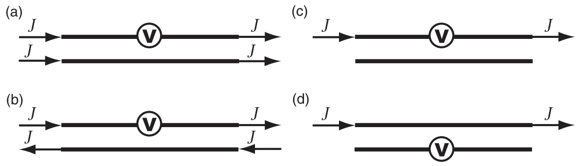

FIG. 2: (a) The same amount of current flows on both layers in the same direction. (b) The same amount of current flows on both layers in the opposite directions. (c) and (d) The current flows only on the front layer. In these experiments the diagonal and Hall resistances are measured at one of the layers indicated by V in a circle.

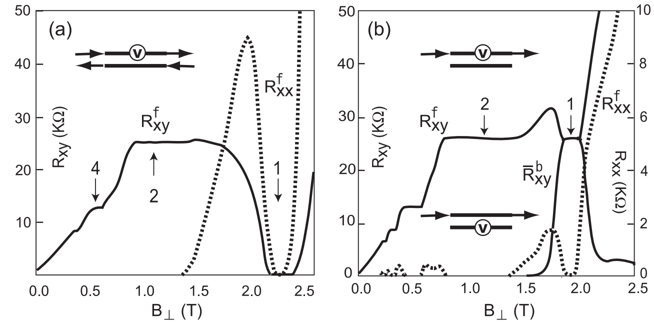

FIG. 3: By feeding different currents into the two layers, the Hall resitance R_{xy} (solid curve) and the longitudinal resistance R_{xx} (dotted curve) are measured on one of the layers (indicated by V in a circle). (a) In the counterflow experiment the opposite amounts of currents are fed to the two layers, where $R^{\rm f}_{xx} = R^{\rm f}_{xy} = 0$ anomalously at $\nu = 1$. Data are taken from Ref.[4]. (b) In the drag experiment the current is fed only to the front layer, where $R^{\rm f}_{xy}$ takes anomalously the same value at $\nu = 1$ and $\nu = 2$. It is also remarkable that $R^{\rm f}_{xy} = \bar{R}^{\rm b}_{xy}$ at $\nu = 1$, where $\bar{R}^{\rm b}_{xy} \equiv E^{\rm b}_y/\mathcal{J}^{\rm f}_x$. Data are taken from Ref.[5].

The condition $E^{\rm f}_y = E^{\rm b}_y$ requires

$$E^{\rm f}_y - E^{\rm b}_y = \frac{2\hbar}{e^2\ell_B^2\rho_0}\left(\frac{\mathcal{J}^{\rm f}_x}{1+\sigma_0} - \frac{\mathcal{J}^{\rm b}_x}{1-\sigma_0} - \frac{2eJ^d_s}{\hbar}\partial_x\vartheta\right) = 0, \tag{6.16}$$

or

$$\partial_x\vartheta = \frac{\hbar}{2eJ^d_s}\left(\frac{\mathcal{J}^{\rm f}_x}{1+\sigma_0} - \frac{\mathcal{J}^{\rm b}_x}{1-\sigma_0}\right). \tag{6.17}$$

Substituting (6.17) into (6.15) we obtain

$$E^{\rm f}_y = E^{\rm b}_y = \frac{\hbar}{e^2\ell_B^2\rho_0}\left(\mathcal{J}^{\rm f}_x + \mathcal{J}^{\rm b}_x\right). \tag{6.18}$$

We conclude that the Hall resistance is given by

$$R^{\rm f}_{xy} \equiv \frac{E^{\rm f}_y}{\mathcal{J}^{\rm f}_x} = \frac{2\pi\hbar}{\nu e^2}\left(1 + \frac{\mathcal{J}^{\rm b}_x}{\mathcal{J}^{\rm f}_x}\right), \tag{6.19a}$$

$$R^{\rm b}_{xy} \equiv \frac{E^{\rm b}_y}{\mathcal{J}^{\rm b}_x} = \frac{2\pi\hbar}{\nu e^2}\left(1 + \frac{\mathcal{J}^{\rm f}_x}{\mathcal{J}^{\rm b}_x}\right) \tag{6.19b}$$

in each layer. We note that both the diagonal and Hall resistances are independent of the imbalance parameter σ_0.

VII. ANOMALOUS BILAYER HALL CURRENTS

We apply these formulas to analyze typical bilayer QH currents [Fig.2], and compare the results with the experimental data[3,4,5]. Though the experiments were carried out at the balanced point ($\sigma_0 = 0$), our results are valid also for imbalanced configurations ($\sigma_0 \neq 0$).

A. Experiment (a)

The same amounts of current are fed to the two layers in the experiment [Fig.2(a)]. Since $\mathcal{J}^{\rm f}_x = \mathcal{J}^{\rm b}_x$, we obtain from (6.17) that

$$\vartheta = \text{constant},$$

and

$$R^{\rm f}_{xy} \equiv \frac{E^{\rm f}_y}{\mathcal{J}^{\rm f}_x} = \frac{4\pi\hbar}{\nu e^2}, \qquad R^{\rm b}_{xy} \equiv \frac{E^{\rm b}_y}{\mathcal{J}^{\rm b}_x} = \frac{4\pi\hbar}{\nu e^2}. \tag{7.1a}$$

This is the standard result of the bilayer QH current.[3,4]

B. Counterflow Experiment (b)

The counterflow experiment [Fig.2(b)] is most interesting. Since $\mathcal{J}^{\rm f}_x = -\mathcal{J}^{\rm b}_x$, we obtain from (6.17) that

$$\vartheta = \frac{\hbar\mathcal{J}^{\rm f}_x}{eJ^d_s}x + \text{constant}. \tag{7.2}$$

and

$$R^{\rm f}_{xy} \equiv \frac{E^{\rm f}_y}{\mathcal{J}^{\rm b}_x} = 0, \qquad R^{\rm b}_{xy} \equiv \frac{E^{\rm b}_y}{\mathcal{J}^{\rm b}_x} = 0. \tag{7.3a}$$

The result is remarkable since it is against the naive picture of the QH effect. Recall that the essential signal of the QH effect is considered to be the development of the plateau. The vanishing of the Hall resistance in the QH regime is a new phenomenon. This anomalous behavior has been observed experimentally by Kellogg et al.[3] and Tutuc et al.[4] at $\nu = 1$, as illustrated in Fig.3(a).

C. Drag Current (c) & (d)

The drag experiment [Figs.2(c) and (d)] is also very interesting, where the current is fed only to the front layer. The Hall resistance is measured in the front layer

[Fig.(c)] and also in the back layer [Figs.(d)]. Since $\mathcal{J}_x^{\mathrm{b}} = 0$, we obtain from (6.17) that

$$\vartheta = \frac{\hbar \mathcal{J}_x^{\mathrm{f}}}{2eJ_s^d} x + \text{constant}. \tag{7.4}$$

In the drag experiment the definition (6.19) for R_{xy}^{b} becomes meaningless since $\mathcal{J}_i^{\mathrm{b}} = 0$. We adopt the definition

$$\bar{R}_{xy}^{\mathrm{b}} \equiv \frac{E_y^{\mathrm{b}}}{\mathcal{J}_x^{\mathrm{f}}} \tag{7.5}$$

with the use of $\mathcal{J}_x^{\mathrm{f}}$. Then, we find

$$R_{xy}^{\mathrm{f}} \equiv \frac{E_y^{\mathrm{f}}}{\mathcal{J}_x^{\mathrm{f}}} = \frac{2\pi\hbar}{\nu e^2}, \qquad \bar{R}_{xy}^{\mathrm{b}} \equiv \frac{E_y^{\mathrm{b}}}{\mathcal{J}_x^{\mathrm{f}}} = \frac{2\pi\hbar}{\nu e^2}. \tag{7.6a}$$

In particular, we have

$$R_{xy}^{\mathrm{f}} = \bar{R}_{xy}^{\mathrm{b}} = \frac{E_y^{\mathrm{f}}}{\mathcal{J}_x^{\mathrm{f}}} = \frac{2\pi\hbar}{e^2} \qquad \text{at} \quad \nu = 1. \tag{7.7}$$

On the other hand, if there is no interlayer coherence, the QH current in the front layer is

$$\mathcal{J}_i^{\mathrm{f}} = \frac{\nu e^2}{4\pi\hbar} \varepsilon_{ij} E_j^{\mathrm{f}} \tag{7.8}$$

at the balance point, and

$$R_{xy}^{\mathrm{f}} = \frac{E_y^{\mathrm{f}}}{\mathcal{J}_x^{\mathrm{f}}} = \frac{4\pi\hbar}{\nu e^2}. \tag{7.9}$$

In particular we have

$$R_{xy}^{\mathrm{f}} = \frac{E_y^{\mathrm{f}}}{\mathcal{J}_x^{\mathrm{f}}} = \frac{2\pi\hbar}{e^2} \qquad \text{at} \quad \nu = 2. \tag{7.10}$$

It is prominent that from (7.7) and (7.10) the Hall resistance is the same at $\nu = 1$ and $\nu = 2$. These theoretical results explain the drag experimental data due to Kellogg et al.[5], as illustrated in Fig.3(b).

VIII. CONCLUSION

In this paper we have analyzed the dynamics of electrons confined to the lowest Landau level based on noncommutative geometry. In ordinary physics the dynamics arises from the kinetic Hamiltonian. In the QH system, however, it arises from the very nature of noncommutative geometry, that is the $W_\infty(N)$ algebra (2.11) satisfied by the bare density $\hat{D}_{\mu\nu}(\boldsymbol{p})$.

As an application we have derived the formula for the electric current. We have found that the Coulomb interaction yields quite complicated contributions through the exchange term to the current. Nevertheless, we reproduce the standard formula for the Hall current in the monolayer QH ferromagnet. However, the Hall current contains the phase current in the bilayer QH ferromagnet. It is a dynamical problem how the phase current flows. We have shown that it flows in such a way that the Hall current behaves anomalously as discovered in recent experiments[3,4,5]. Furthermore, the anomalous Hall resistance is unchanged even if the density imbalance is made between the two layers. These experimental data provide us with another proof of the interlayer phase coherence spontaneously developed in the bilayer system.

APPENDIX A: SU(2) EFFECTIVE HAMILTONIAN

In this appendix we derive the classical Hamiltonian (5.6) from the field-theoretical Hamiltonian (5.5), or $H = H_{\mathrm{C}}^+ + H_{\mathrm{C}}^- + H_{\mathrm{T}} + H_{\mathrm{gate}} + H_{\mathrm{E}}$. Each term is given in terms of the electron density $\rho(\boldsymbol{x})$ and the pseudospin density $P_a(\boldsymbol{x})$ as follows,

$$H_{\mathrm{C}}^+ = \frac{1}{2} \int d^2x d^2y \; V^+(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{x}) \rho(\boldsymbol{y}), \tag{A1a}$$

$$H_{\mathrm{C}}^- = 2 \int d^2x d^2y \; V^-(\boldsymbol{x} - \boldsymbol{y}) P_z(\boldsymbol{x}) P_z(\boldsymbol{y}), \tag{A1b}$$

$$H_{\mathrm{T}} = -\Delta_{\mathrm{SAS}} \int d^2x \; P_x(\boldsymbol{x}), \tag{A1c}$$

$$H_{\mathrm{gate}} = -\Delta_{\mathrm{bias}} \int d^2x \; P_z(\boldsymbol{x}), \tag{A1d}$$

$$H_{\mathrm{E}} = -e \int d^2x \; \left[\varphi^{\mathrm{f}}(\boldsymbol{x}) \rho^{\mathrm{f}}(\boldsymbol{x}) + \varphi^{\mathrm{b}}(\boldsymbol{x}) \rho^{\mathrm{b}}(\boldsymbol{x}) \right], \tag{A1e}$$

where Δ_{bias} is the bias parameter to take care of the charge imbalance made by the gate voltages in (A1d). The Coulomb potentials are

$$V^{\pm}(\boldsymbol{x}) = \frac{e^2}{8\pi\epsilon} \left(\frac{1}{|\boldsymbol{x}|} \pm \frac{1}{\sqrt{|\boldsymbol{x}|^2 + d^2}} \right). \tag{A2}$$

We take the expectation value by the Fock state (3.4). The Coulomb energy is decomposed into the direct and exchange energies[11],

$$\begin{aligned}\langle H_{\mathrm{C}}^+ \rangle = {} & \pi \int d^2p V_{\mathrm{D}}^+(\boldsymbol{p}) \hat{\varrho}(-\boldsymbol{p}) \hat{\varrho}(\boldsymbol{p}) \\ & - \pi \int d^2p V_{\mathrm{X}}^+(\boldsymbol{p}) \left[\mathcal{P}_z(-\boldsymbol{p}) \mathcal{P}_z(\boldsymbol{p}) + \frac{1}{4} \varrho(-\boldsymbol{p}) \varrho(\boldsymbol{p}) \right] \\ & - \pi \sum_{a=x,y} \int d^2p V_{\mathrm{X}}^+(\boldsymbol{p}) \mathcal{P}_a(-\boldsymbol{p}) \mathcal{P}_a(\boldsymbol{p}),\end{aligned} \tag{A3a}$$

$$\begin{aligned}\langle H_{\mathrm{C}}^- \rangle = {} & 4\pi \int d^2p V_{\mathrm{D}}^-(\boldsymbol{p}) \mathcal{P}_z(-\boldsymbol{p}) \mathcal{P}_z(\boldsymbol{p}) \\ & - \pi \int d^2p V_{\mathrm{X}}^-(\boldsymbol{p}) \left[\mathcal{P}_z(-\boldsymbol{p}) \mathcal{P}_z(\boldsymbol{p}) + \frac{1}{4} \varrho(-\boldsymbol{p}) \varrho(\boldsymbol{p}) \right] \\ & + \pi \sum_{a=x,y} \int d^2p V_{\mathrm{X}}^-(\boldsymbol{p}) \mathcal{P}_a(-\boldsymbol{p}) \mathcal{P}_a(\boldsymbol{p}).\end{aligned} \tag{A3b}$$

All other terms are simply converted into the corresponding classical terms,

$$\langle H_{\rm T}\rangle = -2\pi\Delta_{\rm SAS}\mathcal{P}_x(0), \tag{A4}$$
$$\langle H_{\rm gate}\rangle = -2\pi\Delta_{\sf bias}P_z(0), \tag{A5}$$
$$\langle H_{\rm E}\rangle = -e\int d^2q\, e^{-\boldsymbol{q}^2\ell_B^2/4}\left[\varphi^{\rm f}(-\boldsymbol{q})\varrho^{\rm f}(\boldsymbol{q}) + \varphi^{\rm b}(-\boldsymbol{q})\varrho^{\rm b}(\boldsymbol{q})\right]. \tag{A6}$$

We now analyze the ground-state condition. We substitute

$$\varrho(\boldsymbol{x}) = \rho_0, \quad P_a(\boldsymbol{x}) = \frac{1}{2}\rho_0(\sqrt{1-\sigma_0}, 0, \sigma_0) \tag{A7}$$

into the total energy $\mathcal{H} \equiv \langle H\rangle$, and minimize it with respect to σ_0. As a result we obtain the condition

$$\Delta_{\rm bias} = \frac{\sigma_0}{\sqrt{1-\sigma_0^2}}\Delta_{\sf SAS} + \sigma_0\epsilon_{\rm cap} \tag{A8}$$

with (5.10). Eliminating the parameter $\Delta_{\rm bias}$ from the total classical Hamiltonian

$$\mathcal{H} = \langle H_{\rm C}^+\rangle + \langle H_{\rm C}^-\rangle + \langle H_{\rm T}\rangle + \langle H_{\rm gate}\rangle + \langle H_{\rm E}\rangle, \tag{A9}$$

we can rearrange it into

$$\mathcal{H} = \mathcal{H}_{\rm D} + \mathcal{H}_{\rm X} + \mathcal{H}_{\rm T} + \mathcal{H}_{\rm bias} + \mathcal{H}_{\rm E}, \tag{A10}$$

where various terms are given by (5.6) in text.

We proceed to analyze a small fluctuation of the pseudospin field $P_a(\boldsymbol{x})$ around the ground state (A7), which describes the pseudospin wave. We parametrize

$$P_a(\boldsymbol{x}) = \frac{\rho_0}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\pi_a\boldsymbol{n}(\boldsymbol{x}), \tag{A11}$$

where π_a is the Pauli matrix, and

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{x}) &= \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\sigma_0} & \sqrt{1-\sigma_0}\\ \sqrt{1-\sigma_0} & -\sqrt{1+\sigma_0}\end{pmatrix}\begin{pmatrix}\sqrt{1-|\eta(\boldsymbol{x})|^2}\\ \eta(\boldsymbol{x})\end{pmatrix}\\ &\simeq \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\sigma_0} & \sqrt{1-\sigma_0}\\ \sqrt{1-\sigma_0} & -\sqrt{1+\sigma_0}\end{pmatrix}\begin{pmatrix}1\\ \eta(\boldsymbol{x})\end{pmatrix}\end{aligned} \tag{A12}$$

in the linear approximation, with

$$\eta(\boldsymbol{x}) = \frac{\sigma(\boldsymbol{x}) + i\vartheta(\boldsymbol{x})}{2}, \quad \eta^\dagger(\boldsymbol{x}) = \frac{\sigma(\boldsymbol{x}) - i\vartheta(\boldsymbol{x})}{2}. \tag{A13}$$

The pseudospin field (A11) is reduced to the ground state (A7) for $\eta(\boldsymbol{x}) = 0$. Substituting the pseudospin field (A11) into (A10) together with this parametrization, we find

$$\begin{aligned}\mathcal{H} =& \frac{(1-\sigma_0^2)J_s + \sigma_0^2 J_s^d}{2}(\partial_k\sigma)^2\\ &+ \frac{\rho_0}{4}\left[\epsilon_{\rm cap}(1-\sigma_0^2) + \frac{\Delta_{\rm SAS}}{\sqrt{1-\sigma_0^2}}\right]\sigma^2\\ &+ \frac{1}{2}J_s^d(\partial_k\vartheta)^2 + \frac{\rho_0}{4}\frac{\Delta_{\rm SAS}}{\sqrt{1-\sigma_0^2}}\vartheta^2\end{aligned} \tag{A14}$$

up to the second order in $\sigma(\boldsymbol{x})$ and $\vartheta(\boldsymbol{x})$. Their coherence lengths are

$$\begin{aligned}\xi_\vartheta &= 2\ell_B\sqrt{\frac{\pi\sqrt{1-\sigma_0^2}J_s^d}{\Delta_{\rm SAS}}},\\ \xi_\sigma &= 2\ell_B\sqrt{\frac{\pi\left[(1-\sigma_0^2)J_s + \sigma_0^2 J_s^d\right]}{\epsilon_{\rm cap}(1-\sigma_0^2) + \Delta_{\rm SAS}/\sqrt{1-\sigma_0^2}}}.\end{aligned} \tag{A15}$$

It is remarkable that the coherent length ξ_ϑ becomes infinitely large as $\Delta_{\rm SAS} \to 0$, though ξ_σ remains finite due to the capacitance-energy parameter $\epsilon_{\rm cap}$. It follows from (A8) that the condition $\Delta_{\rm bias} < \epsilon_{\rm cap}$ is necessary to obtain the gapless mode.

APPENDIX B: SU(4) EFFECTIVE HAMILTONIAN

We have ignored the spin degree of freedom when we have analyzed the Hall currents in the bilayer QH system in Section VI. In this appendix, including all components of the SU(4) QH system, we derive the effective Hamiltonian and justify this simplification.

There are 15 isospin components,

$$\begin{aligned}S_a &= \frac{1}{2}\Psi^\dagger\sigma_a\Psi(\boldsymbol{x}), \quad P_a = \frac{1}{2}\Psi^\dagger\pi_a\Psi,\\ R_{ab} &= \frac{1}{2}\Psi^\dagger\sigma_a\pi_b\Psi.\end{aligned} \tag{B1}$$

We denote their classical fields as $\mathcal{S}_a$, $\mathcal{P}_a$ and $\mathcal{R}_{ab}$ as in (4.2) and (5.2). The classical Hamiltonian has already been derived[11], where the Coulomb energy density is

$$\begin{aligned}\langle H_{\rm C}^{\rm cl}\rangle =& \pi V_{\rm D}^+(\boldsymbol{p})\varrho(-\boldsymbol{p})\varrho(\boldsymbol{p}) + 4\pi V_{\rm D}^-(\boldsymbol{p})\mathcal{P}_z(-\boldsymbol{p})\mathcal{P}_z(\boldsymbol{p})\\ &- \frac{\pi}{2}V_{\rm X}^d(\boldsymbol{p})[\mathcal{S}_a(-\boldsymbol{p})\mathcal{S}_a(\boldsymbol{p}) + \mathcal{P}_a(-\boldsymbol{p})\mathcal{P}_a(\boldsymbol{p})\\ &\qquad + \mathcal{R}_{ab}(-\boldsymbol{p})\mathcal{R}_{ab}(\boldsymbol{p})]\\ &- \pi V_{\rm X}^-(\boldsymbol{p})[\mathcal{S}_a(-\boldsymbol{p})\mathcal{S}_a(\boldsymbol{p}) + \mathcal{P}_z(-\boldsymbol{p})\mathcal{P}_z(\boldsymbol{p})\\ &\qquad + \mathcal{R}_{az}(-\boldsymbol{p})\mathcal{R}_{az}(\boldsymbol{p})]\\ &- \frac{\pi}{8}V_{\rm X}(\boldsymbol{p})\varrho(-\boldsymbol{p})\varrho(\boldsymbol{p}).\end{aligned} \tag{B2}$$

The Zeeman term, the tunneling term and the bias terms are given by (4.9c), (5.6c) and (5.6d), respectively.

The ground state is given by

$$\begin{aligned}\mathcal{S}_a^{\rm g} &= \frac{1}{2}\delta_{az}, \qquad \mathcal{P}_a^{\rm g} = \frac{1}{2}(\sqrt{1-\sigma_0^2}\delta_{ax} + \sigma_0\delta_{az}),\\ \mathcal{R}_{ab}^{\rm g} &= \frac{1}{2}\delta_{az}(\sqrt{1-\sigma_0^2}\delta_{bx} + \sigma_0\delta_{bz})\end{aligned} \tag{B3}$$

in the imbalanced configuration at σ_0. We analyze a small fluctuation of the isospin field around the ground state. We may parametrize

$$S_a = \frac{\rho_0}{2}\boldsymbol{n}^\dagger\sigma_a\boldsymbol{n}, \quad P_a = \frac{\rho_0}{2}\boldsymbol{n}^\dagger\pi_a\boldsymbol{n}, \quad R_{ab} = \frac{\rho_0}{2}\boldsymbol{n}^\dagger\sigma_a\pi_b\boldsymbol{n}, \tag{B4}$$

where

$$\boldsymbol{n}(\boldsymbol{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} f_+ & 0 & f_- & 0 \\ 0 & f_+ & 0 & f_- \\ f_- & 0 & -f_+ & 0 \\ 0 & f_- & 0 & -f_+ \end{pmatrix} \begin{pmatrix} 1 \\ \eta_{\rm s}(\boldsymbol{x}) \\ \eta_{\rm p}(\boldsymbol{x}) \\ \eta_{\rm r}(\boldsymbol{x}) \end{pmatrix} \tag{B5}$$

in the linear approximation as in (A12), with $f_\pm = \sqrt{1 \pm \sigma_0}$. The phase field $\vartheta_i(\boldsymbol{x})$ and the imbalance field $\sigma_i(\boldsymbol{x})$ are introduced as in (A13) for each component, $i =$s, p, r.

The effective Hamiltonian is derived by substituting (B4) together with (B5) into the classical Hamiltonians and making the derivative expansions. It is found that the pseudospin mode $\eta_{\rm p}(\boldsymbol{x})$ is described precisely by the Hamiltonian (A14). To study the other two modes, we change the variables as

$$\begin{aligned} \eta_s &= \sqrt{\frac{1+\sigma_0}{2}}\eta_1 + \sqrt{\frac{1-\sigma_0}{2}}\eta_2, \\ \eta_r &= \sqrt{\frac{1-\sigma_0}{2}}\eta_1 - \sqrt{\frac{1+\sigma_0}{2}}\eta_2. \end{aligned} \tag{B6}$$

The effective Hamiltonian reads

$$\begin{aligned} \mathcal{H}_{\rm mix} =& \frac{J_s^+ + \sigma_0 J_s^-}{2}[(\partial_k \sigma_1)^2 + (\partial_k \vartheta_1)^2] \\ &+ \frac{\rho_0}{4}\left(\Delta_{\rm Z} + \frac{1}{2}\Delta_{\rm SAS}\sqrt{\frac{1-\sigma_0}{1+\sigma_0}}\right)[\sigma_1^2 + \vartheta_1^2] \\ &+ \frac{J_s^+ - \sigma_0 J_s^-}{2}[(\partial_k \sigma_2)^2 + (\partial_k \vartheta_2)^2] \\ &+ \frac{\rho_0}{4}\left(\Delta_{\rm Z} + \frac{1}{2}\Delta_{\rm SAS}\sqrt{\frac{1+\sigma_0}{1-\sigma_0}}\right)[\sigma_2^2 + \vartheta_2^2] \\ &+ \frac{\rho_0}{4}\Delta_{\rm SAS}(\sigma_1\sigma_2 + \vartheta_1\vartheta_2), \end{aligned} \tag{B7}$$

where

$$J_s^\pm = \frac{1}{2}\left(J_s \pm J_s^d\right) \tag{B8}$$

with (5.18) and (5.19). The two modes η_1 and η_2 are coupled in general, but decoupled for $\Delta_{\rm SAS} = 0$. There exist no gapless modes in this Hamiltonian provided $\Delta_{\rm Z} \neq 0$.

In conclusion, when $\Delta_{\rm SAS} = 0$, there is only one gapless mode, which is the interlayer phase field $\vartheta_{\rm p}$. This justifies that we have neglected all dynamical fields except for the field $\vartheta_{\rm p}$ to analyze the currents in Section VII.

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